Some New Results on Complex Wiener-Itô Integrals

刘勇 北京大学
liuyong@math.pku.edu.cn
Joint with
陈慧萍 (中国科学院数学与系统科学研究院), 陈勇 (江西师范大学)

The 19th International Workshop on Markov Processes and Related Topics
2024.7.23-26 北京师范大学-福建师范大学 福州

Outline

1. Background
   - Wiener-Itô Chaos Decomposition
   - Complex Winer-Itô Decomposition
   - Connection between real and complex chaos decomposition

2. Some new results
   - Kernel Representation Formula from Complex to Real Wiener-Itô Integrals
   - Ergodicity for Complex Ginzburg-Landau Equation with a Complex-valued Space-time White Noise on $\mathbb{T}^2$
   - Berry-Esséen bound for complex Wiener-Itô integral

1. Background
   - Wiener-Itô Chaos Decomposition
   - Complex Winer-Itô Decomposition
   - Connection between real and complex chaos decomposition

2. Some new results
   - Kernel Representation Formula from Complex to Real Wiener-Itô Integrals
   - Ergodicity for Complex Ginzburg-Landau Equation with a Complex-valued Space-time White Noise on $\mathbb{T}^2$
   - Berry-Esséen bound for complex Wiener-Itô integral

- ☐ Probabilistic Model:
  - Input: $(W_t)_{t\in[0,\infty)}$, Brownian Motion, (noise);
  - Output: $F\big((W_t)_{t\in[0,\infty]}\big)$, $\mathbf{E}\Big(F^2\big((W_t)_{t\in[0,\infty)}\big)\Big) < \infty$.
- ☐ Classical Wiener space:
  $W = C_0[0,\infty) = \{f : \text{continuous func. on } [0,\infty), f(0) = 0\}$.

$$(W, \mathcal{B}(W), \mathbf{P}^W)$$

- ☐ Analysis on $L^2(W, \mathcal{B}(W), \mathbf{P}^W)$.

Theorem (Wiener-Itô Chaos Decomposition)

$$L^2(C_0[0,\infty),\mathbf{P}^W) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

$$\mathcal{H}_n = \overline{\mathrm{span}}\Big\{ H_n\Big(\int_0^\infty h(s)\mathrm{d}W(s)\Big), h \in H, \|h\|_{L^2(\mathbb{R}_+)} = 1 \Big\},$$

$$H_n\Big(\int_0^\infty h(s)\mathrm{d}W(s)\Big)$$
$$= n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} h(t_1)\cdots h(t_n)\mathrm{d}W_{t_1}\cdots \mathrm{d}W_{t_n} = I_n(h^{\odot n}).$$

$H_n(x)$: Hermite Polynomials

- Hermite Polynomials, eigenfunctions of $\Delta_{OU} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}$,

$$\Delta_{OU} H_n(x) = -nH_n(x).$$

-

$$\begin{aligned} H_0(x) &= 1; \\ H_1(x) &= x; \\ H_2(x) &= x^2 - 1; \\ H_3(x) &= x^3 - 3x; \\ H_4(x) &= x^4 - 6x^2 + 3; \\ H_5(x) &= x^5 - 10x^3 + 15x; \\ &\cdots \end{aligned}$$

- $\{H_n\}$: orthonormal basis, $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x)$,

$$\langle H_n, H_m \rangle_\mu = n! \delta_{nm}.$$

☐ Wiener (1938): *The Homogeneous Chaos*

Chaos ≈ Randomness, Singularity, Disorder ...

Homogeneous: monomials of some simply random variables

- Generalized harmonic analysis
- Singular signal processes: power spectral analysis
- Generalized space-time Birkhoff ergodic theorem
- More than normal distribution
- Stratonowich multiple integrals

Cameron-Martin's Chaos decomposition

☐ Cameron-Martin (1947): *in the series of Fourier-Hermite...*

- $(C_0[0,1], \mathbf{P}^W)$, $L^2(C_0[0,1], \mathbf{P}^W)$
- $\{\alpha_p\}$ orthonormal basis in $L^2(0,1)$, H_m Hermite polynomial of degree m

$$\Phi_{m,p} = H_m\Big(\int_0^1 \alpha_p(s) \mathrm{d}\omega(s) \Big)$$

- Not connect with Itô multiple integrals

- Itô (1951): *Multiple Wiener Integrals*
  - System of normal random measures $\mathbf{B}$

$$\int \cdots \int \varphi(t_1)\cdots\varphi(t_n)\mathrm{d}\beta(t_1)\cdots\mathrm{d}\beta(t_n)$$
$$= H_n\Big(\int \varphi(s)\mathrm{d}\beta(s)\Big)$$

  - Iterated stochastic integrals
  - Orthogonalizing Wiener's chaos polynomials

STROOCK'S FORMULAR

☐ Stroock (1987) : *Homogeneous chaos revisited*

STROOCK'S FORMULA

$$F \in L^2(C_0[0,\infty), \mathbf{P}^W),$$

$$F = \mathbf{E}[F] + \sum_{p=1}^{\infty} F_p = \mathbf{E}[F] + \sum_{p=1}^{\infty} I_p(f_p), \;\; F_p \in \mathcal{H}_p,$$

$$F \in \mathbb{D}^{n,2}, \quad f_p = \frac{1}{p!}\mathbf{E}[\mathcal{D}^p F] \;\text{ for all }\; p \leq n.$$

$\mathcal{D}$: Malliavin Derivative

Gaussian Hilbert Space

- Probabilistic Model:
  - Input: $(W(\theta))_{\theta\in\mathcal{I}}$, Gaussian field (noise)
  - Output: $F\big((W(\theta))_{\theta\in\mathcal{I}}\big)$, $\mathbf{E}F^2 < \infty$.
- Closeness of linear operation and L^2-convergence of Gaussian r.v.
  - Input: zero-mean Gaussian r.v.: $\mathcal{H} = (W(h))_{h\in\mathfrak{H}}$,

$$\begin{aligned} \mathfrak{H} &: \text{separable Hilbert space} \\ \mathcal{H} &: \text{closed subspace of zero-mean Gaussian r.v.s} \\ \mathcal{H} &\cong \mathfrak{H},\ \mathbf{E}(W(h_1)W(h_2)) = \langle h_1, h_2\rangle_{\mathfrak{H}} \\ (W(h))_{h\in\mathfrak{H}} &: \text{isonormal Gaussian process over } \mathfrak{H} \end{aligned}$$

  - Gaussian-Hilbert space: $(\Omega, \sigma(\mathcal{H}), \mathbf{P})$
  - Analysis on $L^2(\Omega, \sigma(\mathcal{H}), \mathbf{P})$

THEOREM (CHAOS DECOMPOSITION: GAUSSIAN HILBERT SPACE)

$\mathfrak{H}$: *seperable Hilbert space*

$\mathcal{H}$: *closed subspace of zero-mean Gaussian r.v.s*

$\mathcal{H} \cong \mathfrak{H}, \ \mathbf{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathfrak{H}}$

$$L^2(\Omega, \sigma(\mathcal{H}), \mathbf{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$$\begin{aligned}\mathcal{H}_n &= \overline{\mathrm{span}}\{H_n(W(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\} \\ &= \overline{\mathrm{span}}\{I_n(f), \ f \in \mathfrak{H}^{\odot n}\}\end{aligned}$$

Complex isonormal Gaussian process

☐ Probabilistic Model:

- Input: $(W(\theta) = W^{(1)}(\theta) + \mathrm{i}W^{(2)}(\theta))_{\theta\in\mathcal{I}}$, $W^{(1)}(\theta), W^{(2)}(\theta)$ i.i.d. Gaussian r.v. (noise)
- Output: $F^{(1)}\left(\left(W(\theta)\right)_{\theta\in\mathcal{I}}\right) + \mathrm{i}F^{(2)}\left(\left(W(\theta)\right)_{\theta\in\mathcal{I}}\right)$, $\mathbf{E}|F|^2 < \infty$.

☐ Itô (1952): *Complex Multiple Wiener Integrals*

- System of complex normal random measures $\mathbf{M}$
- $H_{p,q}(z,\overline{z})$: complex Hermite polynomial of degree (p,q), $\|f\|_2 = 1$

$$\int \cdots \int f(t_1) \cdots f(t_p) \overline{f(t_{s_1})} \cdots \overline{f(t_{s_q})}$$
$$\mathrm{d}M(t_1) \cdots \mathrm{d}M(t_p) \mathrm{d}\overline{M(s_1)} \cdots \mathrm{d}\overline{M(s_q)}$$
$$= H_{p,q}\Big(\int f(s) \mathrm{d}M(s), \overline{\int f(s) \mathrm{d}M(s)} \Big)$$

- Normal screw line: spectral structure of shift transformation T_t, ergodicity.

- Probabilistic model of cosmic microwave background radiation
- Stochastic complex Ginzburg-Landau equation
- Chandler wobble
- Communication and signal processes
- ...

Example:

$$\zeta_t = \frac{B_1(t) + \mathrm{i}B_2(t)}{\sqrt{2}}, \quad Z : L^2_{\mathbb{C}}(\mathbb{R}^+) \to L^2_{\mathbb{C}}(\Omega, \mathcal{F}, P)$$

$$\begin{aligned} h_{\mathbb{C}}(= u + \mathrm{i}v) \mapsto Z(h_{\mathbb{C}}) &:= \int_0^\infty h_{\mathbb{C}}(t)\mathrm{d}\zeta_t \\ &= \frac{1}{\sqrt{2}}\left(\left[\int_0^\infty u(t)\mathrm{d}B_1(t) + \int_0^\infty v(t)\mathrm{d}B_2(t)\right]\right. \\ &\quad \left. + \mathrm{i}\left[\int_0^\infty v(t)\mathrm{d}B_1(t) - \int_0^\infty u(t)\mathrm{d}B_2(t)\right]\right) \end{aligned}$$

$$\mathrm{E}[Z(h_{\mathbb{C}})] = 0, \quad \mathrm{E}[Z(h_{\mathbb{C}})^2] = 0, \quad \mathrm{E}[|Z(h_{\mathbb{C}})|^2] = \|h_{\mathbb{C}}\|^2_{L^2_{\mathbb{C}}(\mathbb{R}^+)}.$$

$$\mathcal{I}_{p,q}(f) = \int_{\mathbb{R}_+^{p+q}} f(t_1, \dots, t_p; s_1, \dots, s_q)\, \mathrm{d}\zeta_{t_1} \cdots \mathrm{d}\zeta_{t_p} \mathrm{d}\overline{\zeta_{s_1}} \cdots \mathrm{d}\overline{\zeta_{s_q}}.$$

$f \in L^2_{\mathbb{C}}(\mathbb{R}_+^{p+q})$ is respectively symmetric w.r.t. p and q variables.

Definition

$\mathfrak{H}$: separable Hilbert space

$X,\ Y$: i.i.d. real Gaussian isonormal process over $\mathfrak{H}$

$X_{\mathbb{C}},\ Y_{\mathbb{C}}$: complexicfation of $X,\ Y$

$$Z(\mathfrak{h}) = \frac{X_{\mathbb{C}}(\mathfrak{h}) + \mathrm{i}\,Y_{\mathbb{C}}(\mathfrak{h})}{\sqrt{2}},\ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}$$

$$(\Omega, \sigma(X, Y), \mathbf{P})$$

$$L^2_{\mathbb{C}}(\Omega, \sigma(X, Y), \mathbf{P})$$

Generating function of real Hermite polynomial

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

$$\frac{\partial}{\partial x} H_n(x) = nH_{n-1}(x), \quad \nabla^* \cdot H_{n-1}(x) = H_n(x)$$

Generating function of Complex Hermite polynomial

$$\exp\left(-t\bar{t} + t\bar{z} + \bar{t}z\right) = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} H_{p,q}(z,\bar{z})\bar{t}^p t^q.$$

$$\frac{\partial}{\partial z} H_{p,q} = pH_{p-1,q}, \ \frac{\partial}{\partial \bar{z}} H_{p,q} = H_{p,q-1}, \ldots$$

THEOREM (Itô 1952)

$$\mathscr{H}_{m,n}(Z) = \overline{\text{span}}\{J_{m,n}(Z(\mathfrak{h})),\ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}, \|\mathfrak{h}\|_{\mathfrak{H}_{\mathbb{C}}}\}$$

$$L^2_{\mathbb{C}}(\Omega, \sigma(X, Y), P) = \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} \mathscr{H}_{m,n}$$

DEFINITION (Complex Wiener-Itô multiple integrals)

$$\mathbf{J_{m,n}} := \prod_k \frac{1}{\sqrt{2^{m_k+n_k} m_k! n_k!}} H_{m_k,n_k}(\sqrt{2} Z(\mathfrak{e}_k)).$$

$$\mathscr{I}_{m,n}(\text{symm}(\otimes_{k=1}^{\infty} \mathfrak{e}_k^{\otimes m_k}) \otimes \text{symm}(\otimes_{k=1}^{\infty} \bar{\mathfrak{e}}_k^{\otimes n_k})) = \sqrt{\mathbf{m!n!}} \mathbf{J_{m,n}}.$$

$$\mathscr{I}_{m,n}(f) : \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n} \mapsto \mathscr{H}_{m,n}.$$

☐ Chen, L. (2019)

Theorem (Stroock's formula)

Every random variable $F \in L^2(\Omega, \sigma(Z), P)$ *can be expressed by*

$$F = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} I_{p,q}(f_{p,q}),$$

where $f_{p,q} \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$. *If* $F \in \mathbb{D}^{m,2} \bigcap \bar{\mathbb{D}}^{n,2}$ *then*

$$f_{p,q} = \frac{1}{p!q!} \mathbb{E}[D^p \bar{D}^q F], \qquad \forall p \leq m,\ q \leq n.$$

$$f\big(Z(\varphi_1), \cdots, Z(\varphi_m)\big), \quad f \in C_{\uparrow}^{\infty}(\mathbb{C}^m)$$

$$DF = \sum_{i=1}^{m} \partial_i f(Z(\varphi_1), \dots, Z(\varphi_m))\varphi_i, \ \ \bar{D}F = \sum_{i=1}^{m} \bar{\partial}_i f(Z(\varphi_1), \dots, Z(\varphi_m))\bar{\varphi}_i,$$

where

$$\partial_j f = \frac{\partial}{\partial z_j} f(z_1, \dots, z_m) = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} - \mathrm{i}\frac{\partial f}{\partial y_j}\Big), \quad \bar{\partial}_j f = \frac{\partial}{\partial \bar{z}_j} f(z_1, \dots, z_m) = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} + \mathrm{i}\frac{\partial f}{\partial y_j}\Big)$$

are Wirtinger derivative.

THEOREM (Chen & L. 17)

Suppose that $\varphi \in \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n}$ *and* $F = \mathscr{I}_{m,n}(\varphi) = U + \mathrm{i}V$*, There exist real* $u,\ v \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(m+n)}$ *such that*

$$U = \mathcal{I}_{m+n}(u), \quad V = \mathcal{I}_{m+n}(v),$$

where $\mathcal{I}_p(g)$ *is the p-th real Wiener-Itô multiple integral of g with respect to W. And if* $m \neq n$ *then*

$$E[U^2] = E[V^2], \quad E[UV] = (m+n)!\langle u,\ v\rangle_{(\mathcal{H}\oplus\mathcal{H})^{\otimes(m+n)}} = 0.$$

W: Gaussian Hilbert spaces over $\mathfrak{H} \oplus \mathfrak{H}$. "$X \oplus Y$"
$\mathcal{H}_n(W)$: n-th Chaos decomposition of W.

Theorem (Chen & L. 17)

$$\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathcal{H}_k(X)\mathcal{H}_l(Y),$$

$$\mathcal{H}_n(W) + \mathrm{i}\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathscr{H}_{k,l},$$

$$L^2(\Omega, \sigma(Z), P) = \bigoplus_{n=0}^{\infty} (\mathcal{H}_n(W) + \mathrm{i}\mathcal{H}_n(W))$$

$$= \bigoplus_{n=0}^{\infty} \bigoplus_{k+l=n} \mathscr{H}_{k,l} = \bigoplus_{m=0}^{\infty} \bigoplus_{n=0}^{\infty} \mathscr{H}_{m,n}.$$

Where $\mathcal{H}_n(W), \mathcal{H}_n(X)$ and $\mathcal{H}_n(Y)$ are the n-th Wiener-Itô Chaos with respect to W, X and Y respectively.

Theorem (Nualart-Peccati Criterion, 2005)

For $q \geq 2$, $F_n = I_q(f_n)$, $f_n \in \mathfrak{H}^{\odot q}$, $n \geq 1$. $\mathbf{E}(F_n^2) \to 1$. The following 4 conditions are equivalent, as $n \to \infty$,

(i) $F_n \xrightarrow{d} \mathcal{N}(0,1)$;

(ii) $\mathbf{E}(F_n^4) \to 3$;

(iii) $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \to 0$, $r = 0, 1 \cdots, q-1$;

(iv) $\|D(F_n)\|_{\mathfrak{H}}^2 \to q$ in L^2.

Does 4th Moment Theorem hold for complex Wiener-Itô integrals?

THEOREM (Chen & L. 17)

F_k : (m, n)-th complex Wiener-Itô multiple integrals, $m + n \geq 2$, $E[|F_k|^2] \to \sigma^2$ as $k \to \infty$.

(1) *If $m \neq n$, as $k \to \infty$, then*

(i) $(F_k) \xrightarrow{d} \zeta \sim \mathcal{CN}(0, \sigma^2)$;

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to 2\sigma^4$.

THEOREM (CONTINUED)

(2) *If* $m = n$, $E[F_k^2] \to \sigma^2(a + \mathrm{i}b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 < 1$, *then*

(i) $(\mathrm{Re} F_k, \mathrm{Im} F_k) \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{2}\mathrm{C})$, *where* $\mathrm{C} = \begin{bmatrix} 1+a & b \\ b & 1-a \end{bmatrix}$,

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to (a^2 + b^2 + 2)\sigma^4$.

(3) *If* $m = n$, $E[F_k^2] \to \sigma^2(a + \mathrm{i}b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, *then*

(i) $(\mathrm{Re} F_k, \mathrm{Im} F_k) \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{2}\mathrm{C})$

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to 3\sigma^4$

$$\Leftrightarrow$$

(iii) $E[{F_k}^4] \to 3(a + \mathrm{i}b)^2\sigma^4$.

Theorem (Chen & L. 17)

Suppose that $\varphi \in \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n}$ *and* $F = \mathscr{I}_{m,n}(\varphi) = U + \mathrm{i}V$*, There exist real* $u, v \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(m+n)}$ *such that*

$$U = \mathcal{I}_{m+n}(u), \quad V = \mathcal{I}_{m+n}(v),$$

where $\mathcal{I}_p(g)$ *is the p-th real Wiener-Itô multiple integral of g with respect to W. And if* $m \neq n$ *then*

$$E[U^2] = E[V^2], \quad E[UV] = (m+n)!\langle u, v\rangle_{(\mathcal{H}\oplus\mathcal{H})^{\otimes(m+n)}} = 0.$$

☐ Problems:

- Existence result
- The representations of u, v depend on some redundant parameters

1 Background
- Wiener-Itô Chaos Decomposition
- Complex Winer-Itô Decomposition
- Connection between real and complex chaos decomposition

2 Some new results
- Kernel Representation Formula from Complex to Real Wiener-Itô Integrals
- Ergodicity for Complex Ginzburg-Landau Equation with a Complex-valued Space-time White Noise on $\mathbb{T}^2$
- Berry-Esséen bound for complex Wiener-Itô integral

Clarify the relation between complex and real Wiener-Itô integrals.

- Real $\Rightarrow$ complex Wiener-Itô integrals.
- Essential differences between them.

Theorem (Chen, Chen and L., 2024)

$\mathcal{I}_{p,q}(f)$, $f \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$, admits the unique representation

$$\mathcal{I}_{p,q}(f) = I_{p+q}(u) + \mathrm{i} I_{p+q}(v),$$

where $u, v \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(p+q)}$ and $I_{p+q}(\cdot)$ is real $(p+q)$-th Wiener-Itô integral w.r.t the real Gaussian process W over $\mathfrak{H} \oplus \mathfrak{H}$ defined as

$$W(f, g) := X(f) + Y(g) \ \text{for } f, g \in \mathfrak{H}.$$

Representation:

- recursion formula: algorithm
- Generalized Stroock's formula: computable

[1] Stochastic Process. Appl., 167: 104241

$$F=(F_1,\dots,F_d),\ F_i\in \mathrm{D}^{1,2},\ \gamma_F=\big(\langle DF_k,DF_j\rangle_{\mathfrak{H}}\big)_{1\le k,j\le d},$$

$$\det\gamma_F>0,\ \text{a.s.}\Rightarrow \mathrm{Law}(F)\ll \mathrm{Leb}(\mathbb{R}^d).$$

Theorem (Chen ,Chen and L., 2024)

$$F=\mathcal{I}_{p,q}(\eta_{k_1}\otimes\cdots\otimes\eta_{k_p}\otimes\overline{\eta_{j_1}}\otimes\cdots\otimes\overline{\eta_{j_q}}):=F_1+\mathrm{i}F_2.$$

1. $F_1=I_{p+q}(u_{p,q}(\mathbf{k},\mathbf{j})),\quad F_2=I_{p+q}(v_{p,q}(\mathbf{k},\mathbf{j}))$.
2. $\mathrm{Law}(F_1,F_2)\ll \mathrm{Leb}(\mathbb{R}^2)\iff p\neq q$ or $p=q$ and $\exists\, 1\le l\le p$ s.t. $k_l\neq j_l$.

[1] Stochastic Process. Appl., 167: 104241

Stochastic heat equation with dispersion on $\mathbb{T}^d$,

$$\partial_t Z_{-\infty,t} = ((\mathrm{i}+\mu)\Delta - 1)\, Z_{-\infty,t} + \xi,\ t > 0,\ x \in \mathbb{T}^d,$$

where $\mu > 0$, ξ is complex-valued space-time white noise.

A stationary solution is a distribution-valued process

$$Z_{-\infty,t} := \sum_{k\in\mathbb{Z}^d} \mathcal{I}_{1,0}\left(f_{t,k}(\cdot)\right) e^{2\pi \mathrm{i} k\cdot x},$$

where

$$f_{t,k}(s) := \mathbf{1}_{(-\infty,t]}(s) e^{-\left(1+4\pi^2\mu|k|^2+\mathrm{i}4\pi^2|k|^2\right)(t-s)}, \quad s \in \mathbb{R}.$$

Application: Regularity of Stochastic Heat Equation with Dispersion on $\mathbb{T}^d$

Proposition (Chen, Chen and L., 2024)

For every $p \in [1, \infty)$,

$$\sup_{t \in \mathbb{R}} \mathrm{E}\left[\|Z_{-\infty,t}\|^p_{\mathcal{C}^{(1-d/2)-}}\right] < \infty,$$

and for every $p \in [1, \infty)$ and $\alpha \in (0,1)$,

$$\sup_{s<t} \frac{\mathrm{E}\left[\|Z_{-\infty,t} - Z_{-\infty,s}\|^p_{\mathcal{C}^{(1-d/2-\alpha)-}}\right]}{|t-s|^{\alpha p/2}} < \infty.$$

Here, $\mathcal{C}^\alpha$ for $\alpha \in \mathbb{R}$ is the Besov space.

[1] Stochastic Process. Appl., 167: 104241

Complex Ginzburg-Landau Equation with Complex Space-time White Noise on $\mathbb{T}^2$

$$\begin{cases} \partial_t u = (\mathrm{i}+\mu)\Delta u - \nu |u|^{2m} u + \tau u + \xi, \quad t>0, \quad x \in \mathbb{T}^2, \\ u(0,\cdot) = u_0. \end{cases}$$

- $\mu > 0$, $\nu, \tau \in \mathbb{C}$, $\mathrm{Re}\,\nu > 0$, $m \geq 1$ is an integer.
- Dispersion term: $\mathrm{i}\Delta u$; dissipation term: $\mu\Delta u$.
- ξ: complex-valued space-time white noise with regularity $(-2)^-$.
- Quantum field theory: complex-valued $\Phi_2^{2(m+1)}$ measure.
- Hoshino, Inahama and Naganuma, 2017[1]; Hoshino, 2018[2]:

  Regularity structure (Hairer) + Paracontrolled distribution (Gubinelli)

  $\Rightarrow$ local and global well-posedness on $\mathbb{T}^3$.

[1] Electron. J. Probab., 22(104), 1-68

[2] Ann. Inst. Henri Poincaré Probab. Stat., 54(4), 1969-2001

stochastic heat equ. $\begin{cases} \partial_t Z_{0,t} = ((\mathrm{i}+\mu)\Delta - 1)\, Z_{0,t} + \xi, & t>0, \quad x \in \mathbb{T}^2, \\ Z_{0,0} = 0. \end{cases}$

Remainder term $\begin{cases} \partial_t v = [(\mathrm{i}+\mu)\Delta - 1]\, v + \Psi(v, \underline{Z}), & t>0, x \in \mathbb{T}^2, \\ v(0,\cdot) = u_0 \in \mathcal{C}^{-\alpha_0}, \end{cases}$

where

$$\begin{aligned}\Psi\left(v_t, \underline{Z}_t\right) \text{``}=\text{''} & -\nu|v_t + Z_{0,t}|^{2m}(v_t + Z_{0,t}) + (\tau+1)(v_t + Z_{0,t}) \\ := & -\nu \sum_{i=0}^{m+1}\sum_{j=0}^{m} \binom{m+1}{i}\binom{m}{j} v_t^i \overline{v_t}^j Z_{0,t}^{:m+1-i,m-j:} + (\tau+1)(v_t + Z_{0,t}).\end{aligned}$$

- $u = v + Z_{0,\cdot}$ solves the equation

$$\begin{cases} \partial_t u = [(\mathrm{i}+\mu)\Delta - 1]\, u + \Psi(u - Z_{0,t}, \underline{Z}) + \xi, & t>0, \quad x \in \mathbb{T}^2, \\ u(0,\cdot) = u_0 \in \mathcal{C}^{-\alpha_0}. \end{cases}$$

[1] Ann. Probab., 31(4), 1900–1916

☐ Chen, Chen, L. (2024+)

1. Global well-posedness.
   - Regularity of $Z_{0,\cdot}$ and its Wick product.
   - Remainder term v.

$$\left.\begin{array}{r}\text{Fixed point argument} \Rightarrow \text{Local well-posedness}\\ \text{Priori estimate}\end{array}\right\} \Rightarrow \text{Global well-posedness}$$

2. Ergodicity

$$\begin{cases}\text{Existence of invariant measure} \Leftarrow \text{Krylov-Bogoliubov Theorem}\\ \text{Uniquensee of invariant measure} \Leftarrow \text{Generalized coupling}\end{cases}$$

Theorem (Chen, Chen, L. (2024++))

Let $p, q \in \mathbb{N}$ be such that $l := p + q \geq 2$ and $F = I_{p,q}(f)$ with $f \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$. Suppose that $\mathbb{E}[|F|^2] = \sigma^2$, $\mathbb{E}[F^2] = a + \mathrm{i}b$ and $\sigma^2 > \sqrt{a^2 + b^2}$. Let $N \sim \mathcal{N}_2(0, C)$, where $C = \frac{1}{2}\begin{bmatrix} \sigma^2 + a & b \\ b & \sigma^2 - a \end{bmatrix}$. Then

$$d_W(F, N) \leq 4\sqrt{2}\sqrt{\sum_{r=1}^{l-1}\binom{2r}{r}}\frac{\sqrt{\lambda_1}}{\lambda_2}\sqrt{\mathbb{E}[|F|^4] - 2\big(\mathbb{E}[|F|^2]\big)^2 - \big|\mathbb{E}[F^2]\big|^2}$$

$$\leq c_2(p, q, a, b, \sigma)\sqrt{\sum_{0<i+j<l}\|f \otimes_{i,j} h\|^2_{\mathfrak{H}^{\otimes(2(l-i-j))}}},$$

Theorem (continued)

$$d_W(F,N) \geq c_1(a,b,\sigma)\max\left\{\left|\mathbb{E}\left[F^3\right]\right|, \left|\mathbb{E}\left[F^2\overline{F}\right]\right|, \mathbb{E}\left[|F|^4\right] - 2\left(\mathbb{E}\left[|F|^2\right]\right)^2 - \left|\mathbb{E}\left[F^2\right]\right|^2\right\}$$

$$\geq c_1(p,q,a,b,\sigma)\max\left\{\mathbf{1}_{\{p=q\}}\left|\sum_{i=0}^{p}\left\langle f\tilde{\otimes}_{i,p-i}f, h\right\rangle_{\mathfrak{H}^{\otimes(2p)}}\right|,\right.$$

$$\left.\mathbf{1}_{\{p=q\}}\left|\sum_{i=0}^{p}\left\langle f\tilde{\otimes}_{i,p-i}f, f\right\rangle_{\mathfrak{H}^{\otimes(2p)}}\right|, \sum_{0<i+j<l}\|f\otimes_{i,j}h\|^2_{\mathfrak{H}^{\otimes(2(l-i-j))}}\right\},$$

where h is the reverse complex conjugate of f and

$$\lambda_1 = \frac{1}{2}[\sigma^2 + \sqrt{a^2+b^2}],\ \lambda_2 = \frac{1}{2}[\sigma^2 - \sqrt{a^2+b^2}]$$

are the two eigenvalues of the matrix C.

□ Chen H.P., 2024[1] Optimal Rate of Convergence for Vector-valued Wiener-Itô Integral

[1] *ALEA Lat. Am. J. Probab. Math. Stat.* 21(1):179–214, 2024.

Thanks

- Chen Y., Liu Y., (2014) On the eigenfunctions of the complex Ornstein-Uhlenbeck operators. *Kyoto J. Math.*
- Chen Y., (2015) On a nonsymmetric Ornstein-Uhlenbeck semigroup and its generator. *Communications on Stochastic Analysis*
- Chen Y., Liu Y., (2017) On the fourth moment theorem for the complex multiple Wiener-Itô integrals. *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.*
- Chen Y., (2017) Product formula and Independence for Complex Multiple Wiener-Ito Integrals. *Advances in Mathematics (China)*.
- Chen Y., Liu Y., (2019) Complex chaos decomposition revisited. *Acta Mathematica Scientia*

- Chen H.P., Chen Y., Liu Y., (2024) Kernel representation formula: from complex to real Wiener-Itô integrals and vice versa. *Stochastic Processes and Their Applications*
- Chen H.P. (2024) Optimal Rate of Convergence for Vector-valued Wiener-Itô Integral. *ALEA, Lat. Am. J. Proba. Math. Stat.*
- Chen H.P., Chen Y., Liu Y., (2024+) Berry-Esséen bound for complex Wiener-Itô integral. *Submitted, ArXiv: 2407.05353v1*
- Chen H.P., Chen Y., Liu Y., (2024++) Ergodicity for Ginzburg-Landau equation with a complex-valued space-time white noise. *Working paper*

Recursive Formula

$\left\{\eta_k = \eta_k^1 + \mathrm{i}\eta_k^2\right\}_{k\geq 1}$: complete and orthogonal in $\mathfrak{H}_{\mathbb{C}}$, $\|\eta_k\|^2_{\mathfrak{H}_{\mathbb{C}}} = 2$.

$\left\{u_{1,0}(k), v_{1,0}(k)\right\}_{k\geq 1}$: a complete orthonormal basis of $\mathfrak{H} \oplus \mathfrak{H}$ defined as

$$u_{1,0}(k) = \frac{1}{\sqrt{2}}\left(\eta_k^1, -\eta_k^2\right), \quad v_{1,0}(k) = \frac{1}{\sqrt{2}}\left(\eta_k^2, \eta_k^1\right).$$

$u_{p,q}(\mathbf{k},\mathbf{j}), v_{p,q}(\mathbf{k},\mathbf{j}) \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(p+q)}$: recursively defined by

$$u_{0,1}(j) = u_{1,0}(j), \quad v_{0,1}(j) = -v_{1,0}(j),$$

$$u_{p,q}(\mathbf{k},\mathbf{j}) = u_{p-1,q}(\mathbf{k},\mathbf{j})\tilde{\otimes}u_{1,0}(k_p) - v_{p-1,q}(\mathbf{k},\mathbf{j})\tilde{\otimes}v_{1,0}(k_p),$$

$$v_{p,q}(\mathbf{k},\mathbf{j}) = u_{p-1,q}(\mathbf{k},\mathbf{j})\tilde{\otimes}v_{1,0}(k_p) + v_{p-1,q}(\mathbf{k},\mathbf{j})\tilde{\otimes}u_{1,0}(k_p).$$

$\mathcal{D}$ and $\overline{\mathcal{D}}$: complex Malliavin derivative operators w.r.t. Z.
$D = (D_1, D_2)$: real Malliavin derivative operator w.r.t. W.

Lemma (C., Chen and Liu, 2024[1])

$$\mathcal{D} = \frac{D_1 - \mathrm{i} D_2}{\sqrt{2}}, \quad \overline{\mathcal{D}} = \frac{D_1 + \mathrm{i} D_2}{\sqrt{2}}.$$

Theorem (C., Chen and Liu, 2024[1])

$$F = \mathcal{I}_{p,q}(f) = I_{p+q}(u) + \mathrm{i} I_{p+q}(v), \quad f \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q},$$

$$u + \mathrm{i} v = \frac{1}{(p+q)!} 2^{-\frac{p+q}{2}} \left(\mathcal{D} + \bar{\mathcal{D}}, \mathrm{i}\left(\mathcal{D} - \bar{\mathcal{D}}\right)\right)^{\otimes(p+q)} F.$$

[1] Stochastic Process. Appl., 167: 104241

$$L^p(\mathbb{R}^n, dx), \quad L^p(\mathbb{T}^n, dx), \quad p \geq 1.$$

- (Directional) Derivative: $h \in \mathbb{R}^n$

$$\nabla_h f(x) \equiv \lim_{\epsilon \to 0} \frac{f(x+\epsilon h) - f(x)}{\epsilon}.$$

- Gradient:

$$\langle \nabla f(x), h \rangle_{\mathbb{R}^n} = \nabla_h f(x).$$

- Integration by parts: $f \in C_0^\infty, \mathbf{g} = (g_1, \cdots, g_n) \in C_0^\infty$,

$$\int \langle \nabla f(x), \mathbf{g}(x) \rangle_{\mathbb{R}^n} \mathrm{d}x = -\int f(x) \Big(\sum_{k=1}^n \frac{\partial g_k}{\partial x_k} \Big) \mathrm{d}x.$$

- Divergence: $\nabla \cdot \mathbf{g} \equiv \sum_{k=1}^n \frac{\partial g_k}{\partial x_k}$.

- Laplacian: $f, g \in C_0^\infty$

$$\int \langle \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^n} \mathrm{d}x = -\int f(x)(\nabla \cdot \nabla g(x)) \mathrm{d}x,$$

$$\Delta g(x) = \nabla \cdot \nabla g(x) = \sum_{k=1}^{n} \frac{\partial^2 g}{\partial x_k^2}(x).$$

- Fourier transform:
- Taylor's formula:

Analysis: Gaussian measure

$$L^2\big(\mathbb{R}^n, \mu(\mathrm{d}x)\big),$$

$$\mu(\mathrm{d}x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\Big(- \frac{x_1^2 + \cdots + x_n^2}{2} \Big) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

☐ Central Limit Theorem (CLT)

☐ Probabilistic Model:

- Input: $\xi_1 \cdots \xi_n$ i.i.d. $\mathcal{N}(0,1)$ (noise);
- Output: $F(\xi, \cdots, \xi_n)$, $\mathbf{E}(F^2(\xi, \cdots, \xi_n)) < \infty$.

- (Directional) Derivative: $h \in \mathbb{R}^n$

$$\nabla_h f(x) \equiv \lim_{\epsilon \to 0} \frac{f(x + \epsilon h) - f(x)}{\epsilon}.$$

- Gradient:

$$\langle \nabla f(x), h \rangle_{\mathbb{R}^n} = \nabla_h f(x).$$

- Integration by parts: $f \in C^\infty, \mathbf{g} = (g_1, \cdots, g_n) \in C^\infty$,

$$\int \langle \nabla f(x), \mathbf{g}(x) \rangle_{\mathbb{R}^n} \mu(\mathrm{d}x) = \int f \cdot \Big(\sum_{k=1}^{n} (-\frac{\partial}{\partial x_k} + x_k) g_k \Big) \mu(\mathrm{d}x)$$

- Divergence:

$$\nabla^* \cdot = \sum_{k=1}^{n} (-\frac{\partial}{\partial x_k} + x_k).$$

- Ornstein-Uhlenbeck operator:

$$\int \langle \nabla f(x), \nabla g(x)\rangle_{\mathbb{R}^n} \mu(\mathrm{d}x) = \int f(x)(\nabla^* \cdot \nabla g(x))\mu(\mathrm{d}x)$$

$$\Delta_{OU} = -\nabla^* \cdot \nabla = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k} = \Delta - \langle x, \nabla \rangle.$$

- OU processes (equ.): $W = (W_1, \cdots, W_n)$, d-Brownian Motion,

$$dX_t = -X_t dt + \sqrt{2}\mathrm{d}W_t.$$

- OU semigroup:

$$e^{-t\Delta_{OU}} f(x) = E^x(f(X_t)) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(\mathrm{d}y).$$

- Hermite Polynomials, Eigenfunctions of Δ_{OU}

$$\begin{aligned} H_0(x) &= 1; \\ H_1(x) &= x; \\ H_2(x) &= x^2 - 1; \\ H_3(x) &= x^3 - 3x; \\ H_4(x) &= x^4 - 6x^2 + 3; \\ H_5(x) &= x^5 - 10x^3 + 15x; \\ &\cdots \end{aligned}$$

$$\Delta_{OU} H_n = -nH_n.$$

$$\{H_n\} : \text{orthonormal basis, } L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x),$$

$$\langle H_n, H_m \rangle_\mu = n! \delta_{nm}.$$

☐ Probabilistic Model:

- Input: $(W_t)_{t\in[0,1]}$, Brownian Motion, (noise);
- Output: $F\big((W_t)_{t\in[0,1]}\big)$, $\mathbf{E}\Big(F^2\big((W_t)_{t\in[0,1]}\big)\Big) < \infty$.

☐ Classical Wiener space:
$W = C_0[0,1] = \{f :$ continuous func. on $[0,1], f(0) = 0\}$.

$$(W, \mathcal{B}(W), \mathbf{P}^W)$$

☐ Analysis on $L^2(W, \mathcal{B}(W), \mathbf{P}^W)$.

- (Directional) Derivative:
  - Tangent direction:
    - Quasi-invariance, Cameron-Martin Theorem
    - Cameron-Martin space

$$H = \{h \in C_0[0,1], \text{absolute continuous}, \dot{h} \in L^2[0,1]\},$$

$$\|h\|_H = \|\dot{h}\|_{L^2} = \int |\dot{h}(s)|^2 ds.$$

    - Remark: $\mathbf{P}^W(H) = 0$.
  - Malliavin Derivative, Gross-Sobolev Derivative: $h \in H$

$$\mathcal{D}_h F(\omega) = \lim_{\epsilon \to 0} \frac{F(\omega + \epsilon h) - F(\omega)}{\epsilon} \quad \text{in} \quad L^2(\mathbf{P}^W).$$

  - For Example: $f \in \mathcal{S}(\mathbb{R}^n)$, $h \in H$, $F(\omega) = f(\omega_{t_1}, \cdots, \omega_{t_n})$

$$\mathcal{D}_h F(\omega) = \sum_{k=1}^{n} \frac{\partial f(\omega_{t_1}, \cdots, \omega_{t_n})}{\partial x_k} h(t_k).$$

☐ Gradient Operator: random variable $\mathcal{D}F \in H$

$$\mathbf{E}\big(\langle \mathcal{D}F, h\rangle_H\big) = \mathbf{E}(\mathcal{D}_h F)$$

☐ Divergence operator δ: adjoint operator of $\mathcal{D}$ w.r.t. $\mathbf{P}^W$.
random variable $\xi \in H$

$$\mathbf{E}(\delta\xi \cdot F) = \mathbf{E}(\langle \xi, \mathcal{D}F\rangle_H)$$

☐ OU operator: $\mathcal{L} = -\delta\mathcal{D}$.

Theorem (Wiener-Itô Chaos Decomposition)

$$L^2(C_0[0,1],\mathbf{P}^W)=\bigoplus_{n=0}^{\infty}\mathcal{H}_n,$$

$$\mathcal{H}_n=\overline{\mathrm{span}}\Big\{H_n\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big), h\in H, \|h\|_{L^2(0,1)}=1\Big\},$$

$$H_n\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big)$$
$$=n!\int_0^1\int_0^{t_n}\cdots\int_0^{t_2} h(t_1)\cdots h(t_n)\mathrm{d}W_{t_1}\cdots\mathrm{d}W_{t_n}=I_n(h^{\odot n}).$$

Chaos decomposition

$$F \in L^2(C_0[0,1], \mathbf{P}^W),$$

$$F = \mathbf{E}[F] + \sum_{p=1}^{\infty} F_p = \mathbf{E}[F] + \sum_{p=1}^{\infty} I_p(f_p), \;\; F_p \in \mathcal{H}_p,$$

$$f_p = \frac{1}{p!}\mathbf{E}[\mathcal{D}^p F]. \qquad \text{Stroock formula}$$

☐ Wiener (1938): *The Homogeneous Chaos*

Chaos $\approx$ Randomness, Singularity, Disorder ...

monomials of some simply random variables

- Generalized harmonic analysis
- Singular signal processes: power spectral analysis
- Generalized space-time Birkhoff ergodic theorem
- More than normal distribution
- Stratonowich multiple integrals

□ Cameron-Martin (1947): *in the series of Fourier-Hermite...*

- $(C_0[0,1], \mathbf{P}^W)$, $L^2(C_0[0,1], \mathbf{P}^W)$
- $\{\alpha_p\}$ orthonormal basis in $L^2(0,1)$, H_m Hermite polynomial of degree m

$$\Phi_{m,p} = H_m\Big(\int_0^1 \alpha_p(s) \mathrm{d}\omega(s)\Big)$$

- Not connect with Itô multiple integrals

☐ Itô (1951): *Multiple Wiener Integrals*

- System of normal random measures $\mathbf{B}$

$$\int \cdots \int \varphi(t_1) \cdots \varphi(t_n) \mathrm{d}\beta(t_1) \cdots \mathrm{d}\beta(t_n)$$
$$= \frac{1}{\sqrt{2}^n} H_n \Big(\frac{1}{\sqrt{2}} \int \varphi(s) \mathrm{d}\beta(s) \Big)$$

- Iterated stochastic integrals
- Orthogonalizing Wiener's chaos polynomials

Chaos decomposition

- Segal (1956): *Tensor Algebras over Hilbert Spaces*
  - A theory of integration over Hilbert spaces, Quantum Field Theory
  - Harmonic analysis, Fourier-Plancherel transform
  - Algebra of symmetric tensors $\cong_U$ square integrable functions
  - Finitely additive (cylindrical) measure
- Gross (1965): *Abstract Wiener spaces*

Chaos decomposition

☐ Stroock (1987) : *Homogeneous chaos revisited*

Stroock formula

$$F \in L^2(C_0[0,1], \mathbf{P}^W),$$

$$F = \mathbf{E}[F] + \sum_{p=1}^{\infty} F_p = \mathbf{E}[F] + \sum_{p=1}^{\infty} I_p(f_p), \;\; F_p \in \mathcal{H}_p,$$

$$F \in \mathbb{D}^{n,2}, \quad f_p = \frac{1}{p!}\mathbf{E}[\mathcal{D}^p F] \;\; \text{for all} \;\; p \leq n.$$

$$\begin{aligned}
\mathcal{D}_\xi H_n\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big) &= \frac{\mathrm{d}H_n}{dx}\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big)\mathcal{D}_\xi \int h(s)\mathrm{d}W(s) \\
&= nH_{n-1}\Big(\int_0^1 h(s)\mathrm{d}W(s)\Big)\int_0^1 h(s)\dot{\xi}(s)ds.
\end{aligned}$$

Chaos decomposition

$$
\begin{aligned}
& \mathbf{E}\Big((\delta\xi) H_n\big(\int_0^1 h(s)\mathrm{d}W(s)\big)\Big) \\
= \; & \mathbf{E}\Big(\mathcal{D}_\xi H_n\big(\int_0^1 h(s)\mathrm{d}W(s)\big)\Big) \\
= \; & \mathbf{E}\Big(n\int_0^1 \cdots \int_0^1 h(s_1)\cdots h(s_{n-1})\mathrm{d}W(s_1)\cdots \mathrm{d}W(s_{n-1}) \int_0^1 h(s)\dot{\xi}(s)ds\Big) \\
= \; & \mathbf{E}\Big(\int_0^1 \dot{\xi}(s)\mathrm{d}W(s) H_n\big(\int_0^1 h(s)\mathrm{d}W(s)\big)\Big).
\end{aligned}
$$

$$\delta\xi = \int_0^1 \dot{\xi}(s)\mathrm{d}W(s).$$

□ Clark-Ocone formular (1970,1984)

$$F \in \mathbb{D}^{1,2}, \quad F = E(F) + \int_0^1 E[\mathcal{D}_t F | \mathcal{F}_t] \mathrm{d}B_t.$$

□ Hu-Meyer formular (1988):

multiple Stratonovich integrals *v.s.* multiple Wiener-Itô integral

□ Wick product, Fock space

Chaos decomposition

$$\mathrm{d}X_t = h(t)X_t\mathrm{d}B_t, \quad X_0 = 1.$$

$$\exp\Big(\int_0^t h(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t h^2(s)\mathrm{d}s\Big) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\int_0^t h(s)\mathrm{d}B_s, \int_0^t h^2(s)\mathrm{d}s).$$

Picard's iteration

- Probabilistic Model:
  - Input: $(W(\theta))_{\theta\in\mathcal{I}}$, Gaussian field (noise)
  - Output: $F\big((W(\theta))_{\theta\in\mathcal{I}}\big)$, $\mathbf{E}F^2 < \infty$.
- Closeness of linear operation and L^2-convergence of Gaussian r.v.
  - Input: zero-mean Gaussian r.v.: $\mathcal{H} = (W(h))_{h\in\mathfrak{H}}$,

$$\begin{aligned}
\mathfrak{H} &: \text{separable Hilbert space} \\
\mathcal{H} &: \text{closed subspace of zero-mean Gaussian r.v.s} \\
\mathcal{H} &\cong \mathfrak{H},\ \mathbf{E}(W(h_1)W(h_2)) = \langle h_1, h_2\rangle_{\mathfrak{H}} \\
(W(h))_{h\in\mathfrak{H}} &: \text{isonormal Gaussian process over } \mathfrak{H}
\end{aligned}$$

  - Gaussian-Hilbert space: $(\Omega, \sigma(\mathcal{H}), \mathbf{P})$
  - Analysis on $L^2(\Omega, \sigma(\mathcal{H}), \mathbf{P})$

- Janson (1997): *Gaussian Hilbert spaces,* Cambridge
- Malliavin (1997): *Stochastic Analysis,* Springer
- 黄志远，严加安 (1997): 无穷维随机分析引论, 科学出版社
- Sheffield (2007): *Gaussian free fields for mathematicians,* PTRF
- Nualart (1995): *Malliavin calculus and related topics,* Springer
- Hairer (2016): *Advanced Stochastic Analysis.*
- Hu Yaozhong (2017): *Annlysis on Gaussian Spaces,* World Scientific

Theorem (Chaos decomposition: Gaussian Hilbert space)

$\mathfrak{H}$: *seperable Hilbert space*

$\mathcal{H}$: *closed subspace of zero-mean Gaussian r.v.s*

$\mathcal{H} \cong \mathfrak{H}, \ \mathbf{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathfrak{H}}$

$$L^2(\Omega, \sigma(\mathcal{H}), \mathbf{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$$\begin{aligned} \mathcal{H}_n &= \overline{\text{span}}\{H_n(W(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\} \\ &= \overline{\text{span}}\{I_n(f), \ f \in \mathfrak{H}^{\odot n}\} \end{aligned}$$

Derivative, Gradient

$$\xi \in \mathfrak{H},\ \mathcal{D}_\xi W(h) = \lim_{\epsilon \to 0} \frac{W(h) + \epsilon \langle h, \xi \rangle_{\mathfrak{H}} - W(h)}{\epsilon} = \langle h, \xi \rangle_{\mathfrak{H}}$$

$$\mathcal{D} H_n(W(h)) = \frac{\mathrm{d}H_n}{\mathrm{d}x}(W(h))h = nH_{n-1}(W(h))h.$$

Divergence

$$\|h\|_{\mathfrak{H}} = 1,\quad \mathbf{E}(\langle \mathcal{D} H_n(W(h)), \xi \rangle_{\mathfrak{H}}) = \mathbf{E}\big(H_n(W(h))W(\xi)\big).$$

$$\delta \xi = W(\xi).$$

- Remark 1: $\int \frac{\mathrm{d}H_n}{\mathrm{d}x} \mu(\mathrm{d}x) = \int H_n(x) x \mu(\mathrm{d}x)$, $\nabla^* \cdot 1 = x$;
- Remark 2: $W(\xi - \langle \xi, h \rangle)_{\mathfrak{H}} h)$ is independent of $W(\langle \xi, h \rangle)_{\mathfrak{H}} h)$.

- Probabilistic Model:
  - Input: $(W(\theta) = W^{(1)}(\theta) + \mathrm{i}W^{(2)}(\theta))_{\theta\in\mathcal{I}}$, $W^{(1)}(\theta), W^{(2)}(\theta)$ i.i.d. Gaussian r.v. (noise)
  - Output: $F^{(1)}\left(\left(W(\theta)\right)_{\theta\in\mathcal{I}}\right) + \mathrm{i}F^{(2)}\left(\left(W(\theta)\right)_{\theta\in\mathcal{I}}\right)$, $\mathbf{E}|F|^2 < \infty$.

Definition

$\mathfrak{H}$	:	separable Hilbert space
$X,\ Y$	:	i.i.d. real Gaussian isonormal process over $\mathfrak{H}$
$X_{\mathbb{C}},\ Y_{\mathbb{C}}$	:	complexicfation of $X,\ Y$

$$Z(\mathfrak{h}) = \frac{X_{\mathbb{C}}(\mathfrak{h}) + \mathrm{i}\, Y_{\mathbb{C}}(\mathfrak{h})}{\sqrt{2}}, \ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}$$

$$(\Omega, \sigma(X, Y), \mathbf{P})$$

$$L^2_{\mathbb{C}}(\Omega, \sigma(X, Y), \mathbf{P})$$

Example

$$\mathfrak{h} = u + \mathrm{i}v,\ \ u, v \in \mathfrak{H}, \quad X_{\mathbb{C}}(\mathfrak{h}) = X(u) + \mathrm{i}X(v)$$

$$Z(\mathfrak{h}) = \frac{1}{\sqrt{2}}[X(u) - Y(v)] + \frac{\mathrm{i}}{\sqrt{2}}[X(v) + Y(u)]$$

Example

$$\mathfrak{H} = L^2[0,1],\ \ \mathfrak{h}(t) = u(t) + \mathrm{i}v(t)$$

X, Y i.i.d. Brownian motions

$$\begin{aligned} Z(\mathfrak{h}) &= \frac{1}{\sqrt{2}}\int_0^1 (u + \mathrm{i}v)\mathrm{d}(X + \mathrm{i}Y) \\ &= \frac{1}{\sqrt{2}}\Big(\Big[\int_0^1 u\mathrm{d}X + \int_0^1 v\mathrm{d}Y\Big] + \mathrm{i}\Big[\int_0^1 v\mathrm{d}X - \int_0^1 u\mathrm{d}Y\Big]\Big) \end{aligned}$$

- Itô (1953): *Complex Multiple Wiener Integrals*
  - System of complex normal random measures $\mathbf{M}$
  - $H_{p,q}(z,\bar{z})$: complex Hermite polynomial of degree (p,q), $\|f\|_2 = 1$

$$\begin{aligned}&\int\cdots\int f(t_1)\cdots f(t_p)\overline{f(t_{s_1})}\cdots\overline{f(t_{s_q})}\\ &\qquad \mathrm{d}M(t_1)\cdots\mathrm{d}M(t_p)\mathrm{d}\overline{M(s_1)}\cdots\mathrm{d}\overline{M(s_q)}\\ =&H_{p,q}\Big(\int f(s)\mathrm{d}M(s),\overline{\int f(s)\mathrm{d}M(s)}\Big)\end{aligned}$$

  - Normal screw line: spectral structure of shift transformation T_t, ergodicity.

Generating function of real Hermite polynomial

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(x) t^n.$$

$$\frac{\partial}{\partial x} H_n(x) = n H_{n-1}(x), \quad \nabla^* \cdot H_{n-1}(x) = H_n(x)$$

Generating function of Complex Hermite polynomial

$$\exp\left(-t\bar{t} + t\bar{z} + \bar{t}z\right) = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} H_{p,q}(z,\bar{z}) \bar{t}^p t^q.$$

$$\frac{\partial}{\partial z} H_{p,q} = p H_{p-1,q}, \ \frac{\partial}{\partial \bar{z}} H_{p,q} = H_{p,q-1}, \ldots$$

☐ Remark:

$$X \sim \mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x$$

$$X + t \sim \nu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} \mathrm{d}x$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \exp\left(tx - \frac{t^2}{2}\right)$$

$$Z = X + \mathrm{i}Y \sim \mu(\mathrm{d}z\mathrm{d}\bar{z}) = \frac{1}{2\pi} \exp\left(-z\bar{z}\right)\mathrm{d}z\mathrm{d}\bar{z} = \frac{1}{\pi} \exp\left(-(x^2 + y^2)\right)\mathrm{d}x\mathrm{d}y$$

$$Z + t \sim \nu(\mathrm{d}z\mathrm{d}\bar{z}) = \frac{1}{2\pi} \exp\left(-(z-t)(\overline{z-t})\right)\mathrm{d}z\mathrm{d}\bar{z}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(z, \bar{z}) = \exp\left(-t\bar{t} + \bar{z}t + z\bar{t}\right)$$

Theorem (Itô 1953)

$$\mathscr{H}_{m,n}(Z) = \overline{\text{span}}\{J_{m,n}(Z(\mathfrak{h})),\ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}, \|\mathfrak{h}\|_{\mathfrak{H}_{\mathbb{C}}}\}$$

$$L^2_{\mathbb{C}}(\Omega, \sigma(X, Y), P) = \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} \mathscr{H}_{m,n}$$

Definition

$$\mathbf{J_{m,n}} := \prod_k \frac{1}{\sqrt{2^{m_k+n_k} m_k! n_k!}} J_{m_k,n_k}(\sqrt{2} Z(\mathfrak{e}_k)).$$

$$\mathscr{I}_{m,n}(\mathrm{symm}(\otimes_{k=1}^{\infty} \mathfrak{e}_k^{\otimes m_k}) \otimes \mathrm{symm}(\otimes_{k=1}^{\infty} \bar{\mathfrak{e}}_k^{\otimes n_k})) = \sqrt{\mathbf{m!n!}} \mathbf{J_{m,n}}.$$

$$\mathscr{I}_{m,n}(f) : \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n} \mapsto \mathscr{H}_{m,n}.$$

□ Chen, L. (2019)
x is a conformal local martingale and $x_0 = 0$ and $\lambda \in \mathbb{C}$.

$$y(\lambda) = \exp\left\{\bar{\lambda}x + \lambda\bar{x} - |\lambda|^2 \langle x, \bar{x}\rangle\right\}.$$

$$y(\lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\bar{\lambda}^m \lambda^n}{m!n!} J_{m,n}(x, \langle x, \bar{x}\rangle).$$

$$\mathrm{d}y = y(\bar{\lambda}\mathrm{d}x + \lambda \mathrm{d}\bar{x}), \qquad y_0 = 1.$$

$$y_t(\lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \bar{\lambda}^m \lambda^n \sum \int_0^t \int_0^{t_{m+n}} \cdots \int_0^{t_2} \mathrm{d}C_{t_1}\mathrm{d}C_{t_2}\cdots \mathrm{d}C_{t_{m+n}},$$

where $0 < t_1 < t_2 < \cdots < t_{m+n} < t$, $C_t = x_t$ or $C_t = \bar{x}_t$, and the sum is over all n-combinations of $\{1, 2, \ldots, m+n\}$ such that $C_t = \bar{x}_t$.

$$J_{m,n}(x_t, \langle x, \bar{x} \rangle_t) = m!n! \sum \int_0^t \int_0^{t_{m+n}} \cdots \int_0^{t_2} \mathrm{d}C_{t_1} \mathrm{d}C_{t_2} \cdots \mathrm{d}C_{t_{m+n}}.$$

When x is a complex Brownian motion, the right hand side of the above equality is equal to the complex multiple Wiener-Itô integral:

$$\int_0^t \int_0^t \cdots \int_0^t \mathrm{d}x_{t_1} \cdots \mathrm{d}x_{t_m} \mathrm{d}\bar{x}_{t_{m+1}} \cdots \mathrm{d}\bar{x}_{t_{m+n}}.$$

☐ Chen, L. (2014) *Kyoto J. Math.*

1-D complex OU process

$$dZ_t = -\alpha Z_t dt + \sqrt{2\sigma^2} d\zeta_t.$$

$$\alpha = ae^{i\theta} = r + i\Omega,\ \zeta_t = B_1(t) + iB_2(t)$$

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} -a\cos\theta & a\sin\theta \\ -a\sin\theta & -a\cos\theta \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{2\sigma^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} -r & \Omega \\ -\Omega & -r \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{2\sigma^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$$

Invariant measure, Generator of 1-d complex OU process

Invariant measure

$$\mathrm{d}\mu = \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{r(x^2+y^2)}{2\sigma^2}\right\} \mathrm{d}x\mathrm{d}y.$$

Generator

$$\begin{aligned} A &= (-rx+\Omega y)\frac{\partial}{\partial x} + (-\Omega x - ry)\frac{\partial}{\partial y} + \sigma^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &= -r\left[\left(1+\mathrm{i}\frac{\Omega}{r}\right)z\frac{\partial}{\partial z} + \left(1-\mathrm{i}\frac{\Omega}{r}\right)\bar{z}\frac{\partial}{\partial \bar{z}} - \frac{4\sigma^2}{r}\frac{\partial^2}{\partial z \partial \bar{z}}\right] \end{aligned}$$

PROPOSITION (NORMAL OPERATOR)

- $$AA^* = A^*A$$

- $\mathcal{A}_s = \sigma^2\Delta - rx\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}, \quad \mathcal{J} = -\mathrm{i}\Omega(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$

$$\mathcal{A}_s\mathcal{J} = \mathcal{J}\mathcal{A}_s$$

PROPOSITION (Metafune, Pallara, Priola 02, Chen & L. 14)

$$\sigma(A) = \{-(m+n)r + \mathrm{i}(m-n)\Omega, \; m, n = 0, 1, 2\ldots\}$$

THEOREM (Eigenfunction of complex OU operator)

The eigenfunction associated with the eigenvalue $-r(m+n)-\mathrm{i}(m-n)\Omega$ *of* A *is*

$$J_{m,n}(z,\rho)=\begin{cases} z^{m-n}\sum_{r=0}^{n}(-1)^r r!\binom{m}{r}\binom{n}{r}|z|^{2(n-r)}\rho^r, & m\geq n,\\ \bar{z}^{n-m}\sum_{r=0}^{m}(-1)^r r!\binom{m}{r}\binom{n}{r}|z|^{2(m-r)}\rho^r & m<n, \end{cases}$$

$$=\begin{cases} z^{m-n}(-1)^n n! L_n^{m-n}(|z|^2,\rho), & m\geq n,\\ \bar{z}^{n-m}(-1)^m m! L_m^{n-m}(|z|^2,\rho), & m<n. \end{cases}$$

$$J_{m,n}(x,y)=\begin{cases} (-1)^n n!(x+\mathrm{i}y)^{m-n}L_n^{m-n}(x^2+y^2,\rho), & m\geq n,\\ (-1)^m m!(x-\mathrm{i}y)^{n-m}L_m^{n-m}(x^2+y^2,\rho), & m<n, \end{cases}$$

$\rho=\frac{2\sigma^2}{r}$, $L_n^{\alpha}(z,\rho)$ *is Laguerre Polynomial.*

$$f\big(Z(\varphi_1), \cdots, Z(\varphi_m)\big), \quad f \in C_\uparrow^\infty(\mathbb{C}^m)$$

$$DF = \sum_{i=1}^m \partial_i f(Z(\varphi_1), \ldots, Z(\varphi_m))\varphi_i, \ \bar{D}F = \sum_{i=1}^m \bar{\partial}_i f(Z(\varphi_1), \ldots, Z(\varphi_m))\bar{\varphi}_i,$$

where

$$\partial_j f = \frac{\partial}{\partial z_j} f(z_1, \ldots, z_m), \quad \bar{\partial}_j f = \frac{\partial}{\partial \bar{z}_j} f(z_1, \ldots, z_m), \quad j = 1, \ldots, m$$

are Wirtinger derivative.

☐ Chen, L. (2019)

Theorem (Stroock's formula)

Every random variable $F \in L^2(\Omega, \sigma(Z), P)$ *can be expressed by*

$$F = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} I_{p,q}(f_{p,q}),$$

where $f_{p,q} \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$. *If* $F \in \mathbb{D}^{m,2} \bigcap \bar{\mathbb{D}}^{n,2}$ *then*

$$f_{p,q} = \frac{1}{p!q!} \mathbb{E}[D^p \bar{D}^q F], \qquad \forall p \le m,\ q \le n.$$

☐ Chen Y. (2017) *Adv.Math (China)*

THEOREM (PRODUCT FORMULA)

$f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$, $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$,

$$I_{a,b}(f) I_{c,d}(g) = \sum_{i=0}^{a\wedge d} \sum_{j=0}^{b\wedge c} \binom{a}{i}\binom{d}{i}\binom{b}{j}\binom{c}{j} i! j!\, I_{a+c-i-j,b+d-i-j}(f \otimes_{i,j} g).$$

$$\begin{aligned} & f \otimes_{i,j} g \\ = & \sum_{l_1,\dots,l_{i+j}=1}^{\infty} \langle f,\ e_{l_1} \otimes \cdots \otimes e_{l_i} \otimes \bar{e}_{l_{i+1}} \otimes \cdots \otimes \bar{e}_{l_{i+j}} \rangle \otimes \\ & \langle g,\ e_{l_{i+1}} \otimes \cdots \otimes e_{l_{i+j}} \otimes \bar{e}_{l_1} \otimes \cdots \otimes \bar{e}_{l_i} \rangle, \end{aligned}$$

by convention, $f \otimes_{0,0} g = f \otimes g$ denotes the tensor product of f and g.

☐ Hoshino M., Inahama Y. and Naganuma N.,
Stochastic complex Ginzburg-Landau equation with space-time white noise.
Electron. J. Probab. 22, paper no. 104, 68 pp. (2017)

- f, g non-symmetric

□ Chen, L. (2019)

Definition

For $(m,n) \geq 0$, $\pi_{m,n}$ denotes the orthogonal projection of $L^2(\Omega)$ onto $\mathscr{H}_{m,n}$, and $\pi_{\leq(m,n)}$ the orthogonal projection of $L^2(\Omega)$ onto $\bigoplus_{i=0}^{m}\bigoplus_{j=0}^{n}\mathscr{H}_{i,j}$. For any $h_1,\dots,h_{m+n} \in \mathfrak{H}$, the Wick product $:Z(h_1)\dots Z(h_m)\overline{Z(h_{m+1})}\dots\overline{Z(h_{m+n})}:$ is given by

$$:Z(h_1)\dots Z(h_m)\overline{Z(h_{m+1})}\dots\overline{Z(h_{m+n})}: := \pi_{m,n}\big(Z(h_1)\dots Z(h_m)\overline{Z(h_{m+1})}\dots\overline{Z(h_{m+n})}\big).$$

Define the general Wick product by

$$\xi \diamond \eta = \pi_{m+p,n+q}(\xi\eta),$$

if $\xi \in \mathscr{H}_{m,n}$ and $\eta \in \mathscr{H}_{p,q}$, and extend $\diamond$ by bilinearity to a binary operator on the finite order chaos space $\overline{\mathcal{P}}_*(\mathfrak{H}) = \sum_{m,n=0}^{\infty}\mathscr{H}_{m,n}$.

□ Janson (1997), Feynman diagram

☐ Chen, L. (2019)

Definition

Take a complete orthonormal system $\{e_k\}$ in $\mathfrak{H}$. Let $f \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$ and consider the following random variable:

$$S^n_{p,q}(f) = \sum_{i_1,\ldots,i_p=1}^{n} \sum_{l_1,\ldots,l_q=1}^{n} \langle f, (e_{i_1} \hat{\otimes} \ldots \hat{\otimes} e_{i_p}) \otimes (\bar{e}_{l_1} \hat{\otimes} \ldots \hat{\otimes} \bar{e}_{l_q}) \rangle Z(e_{i_1}) \ldots Z(e_{i_p}) \overline{Z(e_{l_1})} \ldots \overline{Z(e_{l_q})}.$$

If the limit in probability of $S^n_{p,q}(f)$ exists as $n \to \infty$, one calls f is Stratonovich integrable. The limit is called the multiple Stratonovich integral of f and is denoted by $S_{p,q}(f)$.

Definition

Suppose that $0 \le k \le p \wedge q$. Denote that

$$\mathrm{Tr}^{k,n} f = \sum_{i_1,\dots,i_{p+q-k}=1}^{n} \langle f, (e_{i_1} \hat{\otimes} \dots \hat{\otimes} e_{i_k} \hat{\otimes} \dots \hat{\otimes} e_{i_p}) \otimes (\bar{e}_{i_1} \hat{\otimes} \dots \hat{\otimes} \bar{e}_{i_k} \hat{\otimes} \bar{e}_{i_{p+1}} \hat{\otimes} \dots \hat{\otimes} \bar{e}_{i_{p+q-k}}) \rangle$$
$$\times (e_{i_{k+1}} \hat{\otimes} \dots \hat{\otimes} e_{i_p}) \otimes (\bar{e}_{i_{p+1}} \hat{\otimes} \dots \hat{\otimes} \bar{e}_{i_{p+q-k}}).$$

If $\mathrm{Tr}^{k,n} f$ converges in $\mathfrak{H}^{\odot(p-k)} \otimes \mathfrak{H}^{\odot(q-k)}$ as $n \to \infty$, then one says that f has a trace of order k and the limit is denoted by $\mathrm{Tr}^k f$.

Theorem (Hu-Meyer formula)

Suppose $f \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$. If the traces of order k of f exist for all $k \leq (p \wedge q)$, then f is Stratonovich integrable and

$$S_{p,q}(f) = \sum_{k=0}^{p\wedge q} k! \binom{p}{k} \binom{q}{k} I_{p-k,\, q-k}(\mathrm{Tr}^k f); \tag{1}$$

$$I_{p,q}(f) = \sum_{k=0}^{p\wedge q} (-1)^k k! \binom{p}{k} \binom{q}{k} S_{p-k,\, q-k}(\mathrm{Tr}^k f). \tag{2}$$

Theorem (Chen & L. 17)

Suppose that $\varphi \in \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n}$ *and* $F = \mathscr{I}_{m,n}(\varphi) = U + \mathrm{i}V$, *There exist real* $u,\, v \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(m+n)}$ *such that*

$$U = \mathcal{I}_{m+n}(u), \quad V = \mathcal{I}_{m+n}(v),$$

where $\mathcal{I}_p(g)$ *is the p-th real Wiener-Itô multiple integral of g with respect to W. And if* $m \neq n$ *then*

$$E[U^2] = E[V^2], \quad E[UV] = (m+n)!\langle u,\, v\rangle_{(\mathcal{H}\oplus\mathcal{H})^{\otimes(m+n)}} = 0.$$

W: Gaussian Hilbert spaces over $\mathfrak{H} \oplus \mathfrak{H}$. "$X \oplus Y$"
$\mathcal{H}_n(W)$: n-th Chaos decomposition of W.

Theorem (Chen & L. 17)

$$\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathcal{H}_k(X)\mathcal{H}_l(Y),$$

$$\mathcal{H}_n(W) + \mathrm{i}\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathscr{H}_{k,l},$$

$$L^2(\Omega, \sigma(Z), P) = \bigoplus_{n=0}^{\infty} \left(\mathcal{H}_n(W) + \mathrm{i}\mathcal{H}_n(W)\right)$$

$$= \bigoplus_{n=0}^{\infty} \bigoplus_{k+l=n} \mathscr{H}_{k,l} = \bigoplus_{m=0}^{\infty} \bigoplus_{n=0}^{\infty} \mathscr{H}_{m,n}.$$

Where $\mathcal{H}_n(W), \mathcal{H}_n(X)$ and $\mathcal{H}_n(Y)$ are the n-th Wiener-Itô Chaos with respect to W, X and Y respectively.

THEOREM (NUALART-PECCATI CRITERION, 2005)

For $q \geq 2$, $F_n = I_q(f_n)$, $f_n \in \mathfrak{H}^{\odot q}$, $n \geq 1$. $\mathbf{E}(F_n^2) \to 1$. The following 4 conditions are equivalent, as $n \to \infty$,

(i) $F_n \xrightarrow{d} \mathcal{N}(0,1)$;

(ii) $\mathbf{E}(F_n^4) \to 3$;

(iii) $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \to 0$, $r = 0, 1 \cdots, q-1$;

(iv) $\|D(F_n)\|_{\mathfrak{H}}^2 \to q$ in L^2.

Does 4th Moment Theorem hold for complex Wiener-Itô integrals?

THEOREM (Chen & L. 17)

F_k : (m, n)-th complex Wiener-Itô multiple integrals, $m + n \geq 2$, $E[|F_k|^2] \to \sigma^2$ as $k \to \infty$.

(1) *If $m \neq n$, as $k \to \infty$, then*

(i) $(F_k) \xrightarrow{d} \zeta \sim \mathcal{CN}(0, \sigma^2)$;

$\Leftrightarrow$

(ii) $E[|F_k|^4] \to 2\sigma^4$.

Theorem (Continued)

(2) *If* $m = n$, $E[F_k^2] \to \sigma^2(a + ib)$, $a, b \in \mathbb{R}$, $a^2 + b^2 < 1$, *then*

(i) $(\mathrm{Re}F_k, \mathrm{Im}F_k) \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{2}\mathsf{C})$, *where* $\mathsf{C} = \begin{bmatrix} 1+a & b \\ b & 1-a \end{bmatrix}$,

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to (a^2 + b^2 + 2)\sigma^4$.

(3) *If* $m = n$, $E[F_k^2] \to \sigma^2(a + ib)$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, *then*

(i) $(\mathrm{Re}F_k, \mathrm{Im}F_k) \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{2}\mathsf{C})$

$$\Leftrightarrow$$

(ii) $E[|F_k|^4] \to 3\sigma^4$

$$\Leftrightarrow$$

(iii) $E[{F_k}^4] \to 3(a + ib)^2\sigma^4$.

- Chen, L. (2019)

THEOREM (CLARK-OCONE FORMULA)

Suppose that $\{Z_t, t \geq 0\}$ *is a complex one-dimensional Brownian motion. If* $F \in \mathbb{D}^{1,2} \bigcap \bar{\mathbb{D}}^{1,2}$, *then*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}(D_t F | \mathcal{F}_t) \mathrm{d} Z_t + \int_0^\infty \mathbb{E}(\bar{D}_t F | \mathcal{F}_t) \mathrm{d} \bar{Z}_t, \quad (3)$$

where the integral is a divergence integral.

Definition (Chen, L. 2019)

Complex Ornstein-Uhlenbeck operators are defined as

$$\mathsf{L} = \delta D, \qquad \bar{\mathsf{L}} = \bar{\delta}\bar{D}.$$

Proposition (Chen, L. 2019)

Suppose that $I_{m,n}(f)$ is the complex Wiener-Itô integral of f with respect to Z for $f \in \mathfrak{H}^{\odot m} \otimes \mathfrak{H}^{\odot n}$. Then one has that

$$\mathsf{L}(I_{m,n}(f)) = mI_{m,n}(f), \qquad \bar{\mathsf{L}}(I_{m,n}(f)) = nI_{m,n}(f). \tag{4}$$

Definition

Fix a $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The OU semigroup is the one-parameter semigroup $\{T_t : t \geq 0\}$ of contraction operators on $L^2(\Omega, \sigma(Z), P)$ defined by

$$T_t(F) = \sum_{m,n=0}^{\infty} e^{-[(m+n)\cos\theta + \mathrm{i}(m-n)\sin\theta]t} I_{m,n}(f_{m,n}),$$

where F is given by $F = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_{m,n}(f_{m,n})$ with $f_{m,n} \in \mathfrak{H}^{\odot m} \otimes \mathfrak{H}^{\odot n}$.
It is clear that the infinitesimal generator of the semigroup $\{T_t\}$ is given by

$$L_\theta = -(e^{\mathrm{i}\theta}\mathsf{L} + e^{-\mathrm{i}\theta}\bar{\mathsf{L}}).$$

PROPOSITION (MEHLER SEMIGROUP (CHEN Y. 2015))

Let $r = e^{i\theta}$ *and* $Z' = \{Z'(h) : h \in \mathfrak{H}\}$ *be an independent copy of* Z*. Then, for any* $F \in L^2(\Omega)$*,*

$$T_t(F)(Z) = \mathbb{E}_{Z'}[F(e^{-rt}Z + \sqrt{1 - e^{-2t\cos\theta}}Z')], \qquad t \geq 0.$$

PROPOSITION (HYPERCONTRACTIVITY (CHEN Y. 2015))

For the fixed $t \geq 0$ *and* $p > 1$*, set* $q(t) = e^{2t\cos\theta}(p-1) + 1$*. Then*

$$\|T_t F\|_{q(t)} \leq \|F\|, \qquad \forall F \in L^p(\Omega).$$

Proposition ((Chen, L. (2019)))

Suppose that $F = I_{m,n}(f)$ *with* $f \in \mathfrak{H}^{\odot m} \otimes \mathfrak{H}^{\odot n}$, *then one has that*

$$\mathbb{E}[|F|^4] = \frac{1}{m}\mathbb{E}\big[2\left\|DF\right\|_{\mathfrak{H}}^2 \times |F|^2 + \langle DF,\, D\bar{F}\rangle_{\mathfrak{H}} \times \bar{F}^2\big].$$

Proposition ((Chen, L. (2019)))

$$\begin{aligned}
&c_1(m,n)\Big(\sum_{0<i+j<l} \|f \otimes_{i,j} h\|^2_{\mathfrak{H}^{\otimes(2(l-r))}} + \sum_{0<i+j<l'} \|f \otimes_{i,j} f\|^2_{\mathfrak{H}^{\otimes(2(l'-r))}}\Big) \\
&\leq \mathbb{E}[|F|^4] - 2\big(\mathbb{E}[|F|^2]\big)^2 - \big|E[F^2]\big|^2 \\
&\leq c_2(m,n)\Big(\sum_{0<i+j<l} \big\|f\tilde{\otimes}_{i,j}h\big\|^2_{\mathfrak{H}^{\otimes(2(l-r))}} + \sum_{0<i+j<l'} \big\|f\tilde{\otimes}_{i,j}f\big\|^2_{\mathfrak{H}^{\otimes(2(l'-r))}}\Big)
\end{aligned}$$

Question

$$\mathrm{Var}(\|DI_{m,n}(f)\|^2_{\mathfrak{H}}) + \mathrm{Var}(\left\|\bar{D}I_{m,n}(f)\right\|^2_{\mathfrak{H}}) + \mathrm{Var}(\langle DI_{m,n}(f),\ D\overline{I_{m,n}(f)}\rangle_{\mathfrak{H}})$$
$$\leq c(m,n)\Big(\mathbb{E}[|F|^4] - 2\big(\mathbb{E}[|F|^2]\big)^2 - \left|E[F^2]\right|^2\Big)$$
$$\leq ????$$

Remark: Nourdin, Peccati 2012

For Real multiple Wiener-Itô integrals, $G \in \mathcal{I}_{\amalg}(\})$, $q \geq 2$,

$$\mathrm{Var}\big(\frac{1}{q}\|DG\|^2_{\mathfrak{H}_{\mathbb{R}}}\big) \leq \frac{q-1}{3q}\big(E[G^4] - 3E[G^2]^2\big) \leq (q-1)\mathrm{Var}\big(\frac{1}{q}\|DG\|^2_{\mathfrak{H}_{\mathbb{R}}}\big).$$

□ Limit Theorem
Major P. (1981,2014) : *Multiple Wiener-Itô Integrals.* Lecture Notes in Mathematics. 849, Springer

☐ Chen, L. (2020) Complex isotropic Gaussian random fields on S^1

- Bourgain (1994)

$$\mathrm{i}u_t + u_{xx} + u|u|^{p-2} = 0, \quad u(x,0) = \varphi(x).$$

$$\varphi_{a,\omega}(x) = a + \sum_{j\in\mathbb{Z};j\neq 0} \frac{g_j(\omega)}{j} e^{\mathrm{i}2\pi jx}, \ a \in \mathbb{C}, g_j \text{ i.i.d. complex Gaussian}$$

- Stochastic complex Ginzburg-Landau equation
- Cosmic Microwave Background (CMB) radiation

$\{\lambda_k, k \in \mathbb{Z}\}$ such that $\lambda_k > 0$ and

$$\sum_{k=-\infty}^{\infty} \lambda_k < \infty.$$

Let $a_k \sim \mathcal{CN}(0, \lambda_k)$ independent complex centered Gaussian r.v. system.

$$T(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{\mathrm{i}k\theta}, \qquad \theta \in S^1.$$

$$\mathrm{Cov}(T(\theta), T(\theta')) = E[T(\theta)\overline{T(\theta')}] = \sum_{k=-\infty}^{\infty} \lambda_k e^{\mathrm{i}k(\theta-\theta')}, \quad \theta, \theta' \in S^1.$$

☐ Model: F: a complex-valued function, subordinated field $F(T)$:

$$F[T](\theta) := F(T(\theta)), \quad \theta \in S^1.$$

$$\tilde{a}_k(F) = \frac{1}{2\pi} \int_0^{2\pi} F(T(\theta)) e^{-\mathrm{i}\theta k} \mathrm{d}\theta.$$

☐ Problem: establish sufficient conditions for the following CLT to hold: as $k \to \infty$,

$$\frac{\tilde{a}_k(F)}{\sqrt{\mathbb{E}\left[\,|\tilde{a}_k(F)|^2\right]}} \overset{law}{\to} \mathcal{CN}(0,1). \tag{5}$$

☐ working ...

Thanks

Thanks

W: Gaussian Hilbert spaces over $\mathfrak{H} \oplus \mathfrak{H}$.
$\mathcal{H}_n(W)$: n-th Chaos decomposition of W.

THEOREM (Chen & L. 15)

Suppose that $\|f\|_{\mathfrak{H}}^2 + \|g\|_{\mathfrak{H}}^2 = 1$,

$$H_n(X(f) + Y(g)) = \sum_{l=0}^{n} \binom{n}{l} \|f\|^l \|g\|^{n-l} H_l\Big(\frac{X(f)}{\|f\|}\Big) H_{n-l}\Big(\frac{Y(g)}{\|g\|}\Big),$$

$$H_l(X(f))H_{n-l}(Y(g)) = \sum_k \mathrm{M}_{l,k}^{-1} H_n\big(\cos\theta_k X(f) + \sin\theta_k Y(g)\big).$$

$$\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathcal{H}_k(X)\mathcal{H}_l(Y).$$

Theorem (Chen & L. 17)

Suppose that $\|f\|_{\mathfrak{H}}^2 + \|g\|_{\mathfrak{H}}^2 = 1$, $\left\|\tilde{f}\right\|_{\mathfrak{H}}^2 + \|\tilde{g}\|_{\mathfrak{H}}^2 = 1$.

$$H_n\big(X(f) + Y(g)\big) + \mathrm{i}H_n\big(X(\tilde{f}) + Y(\tilde{g})\big) = \sum_{k=0}^{n} d_k\big(J_{k,n-k}(Z(\mathfrak{h})) + \mathrm{i}J_{k,n-k}(Z(\tilde{\mathfrak{h}}))\big),$$

where $\mathfrak{h} = \sqrt{2}e^{\mathrm{i}\theta}(f - \mathrm{i}g)$, $\tilde{\mathfrak{h}} = \sqrt{2}e^{\mathrm{i}\theta}(\tilde{f} - \mathrm{i}\tilde{g})$,

$$d_k = \frac{1}{2^n}\sum_{r+s=k}(-1)^s\sum_{l=0}^{n}\binom{n}{l}\binom{l}{r}\binom{n-l}{s}(\cos\theta)^l(\mathrm{i}\cdot\sin\theta)^{n-l}.$$

Theorem (Continued)

Suppose that $\mathfrak{H}_{\mathbb{C}} \ni \mathfrak{h}$ *with* $\|\mathfrak{h}\|_{\mathfrak{H}_{\mathbb{C}}} = \sqrt{2}$,

$$J_{k,n-k}(Z(\mathfrak{h})) = \sum_{i=0}^{n} \tilde{c}_i H_n(X(f_i) + Y(g_i)),$$

$f_i + \mathrm{i} g_i = \frac{1}{\sqrt{2}} e^{\mathrm{i}\theta_i} \bar{\mathfrak{h}}$, *and*

$$\tilde{c}_i = \sum_{j=0}^{n} \mathrm{M}_{j,i}^{-1} \mathrm{i}^{n-k} \sum_{r+s=j} \binom{k}{r} \binom{n-k}{s} (-1)^{n-k-s}.$$

$$\mathcal{H}_n^{\mathbb{C}}(W) := \mathcal{H}_n(W) + \mathrm{i}\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathscr{H}_{k,l}(Z) = H_{\mathbb{C}}^{:n:}.$$

THEOREM (Chen & L. 17)

Suppose that $\varphi \in \mathfrak{H}_{\mathbb{C}}^{\odot m} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot n}$ *and* $F = \mathscr{I}_{m,n}(\varphi) = U + \mathrm{i}V$*, There exist real* $u,\, v \in (\mathfrak{H} \oplus \mathfrak{H})^{\odot(m+n)}$ *such that*

$$U = \mathcal{I}_{m+n}(u), \quad V = \mathcal{I}_{m+n}(v),$$

where $\mathcal{I}_p(g)$ *is the p-th real Wiener-Itô multiple integral of g with respect to W. And if* $m \neq n$ *then*

$$E[U^2] = E[V^2], \quad E[UV] = (m+n)!\langle u,\, v\rangle_{(\mathcal{H}\oplus\mathcal{H})^{\otimes(m+n)}} = 0.$$

Remark

If $\rho = 1$, $J_{m,n}(z, 1)$ is called the Hermite polynomials of complex variables by K. Itô. We name $J_{m,n}(z, \rho)$ the Hermite-Laguerre-Itô Polynomials.

The first few Hermite-Laguerre-Itô polynomials are

$$J_{m,0} = z^m, \quad J_{0,n} = \bar{z}^n,$$
$$J_{1,1} = |z|^2 - \rho, \quad J_{2,1} = z(|z|^2 - 2\rho), \quad J_{3,1} = z^2(|z|^2 - 3\rho), \ldots$$
$$J_{1,2} = \bar{z}(|z|^2 - 2\rho), \; J_{2,2} = |z|^4 - 4\rho|z|^2 + 2\rho^2, \; J_{3,2} = z(|z|^4 - 6\rho|z|^2 + 6\rho^2), \ldots$$
$$\ldots$$

Remark

Ismail, Simeonov. (2014) *Proceedings of the AMS*
Ismail. (2015) *Transactions of the AMS*

- Key point: $\mathcal{A}_s\mathcal{J} = \mathcal{J}\mathcal{A}_s$, common eigenfunctions. Solving β_k

$$\begin{cases} J_{m,n}(x,y) &= \sum_{k=0}^{l} \beta_k H_k(x, \frac{\rho}{2}) H_{l-k}(y, \frac{\rho}{2}) \\ J_{m,n}(x,y) &= -\mathrm{i}\lambda\Omega J_{m,n}(x,y) \\ \mathsf{M}\big(-\mathrm{i}(m-n)\big)\vec{\beta} &= 0 \end{cases}$$

- Complex creation and annihilation operator

$$(\partial^*\phi)(z) = -\frac{\partial}{\partial\bar{z}}\phi(z) + \frac{z}{\rho}\phi(z), \quad (\bar{\partial}^*\phi)(z) = -\frac{\partial}{\partial z}\phi(z) + \frac{\bar{z}}{\rho}\phi(z).$$

$$\begin{aligned} J_{0,0}(z,\rho) &= 1 \\ J_{m,n}(z,\rho) &= \rho^{m+n}(\partial^*)^m(\bar{\partial}^*)^n 1. \end{aligned}$$

☐ Chen, Ge, Xiong, Xu (2016), *J. Math.Phys.*

$$\mathrm{d}Z_t = -e^{\mathrm{i}\theta} Z_t \mathrm{d}t + \sqrt{2\cos\theta}\,\mathrm{d}\zeta_t$$

Entropy Production Rate

$$e_p(t)(\omega) = \frac{1}{t}\log\frac{\mathrm{d}\mathbf{P}_{[0,t]}}{\mathrm{d}\mathbf{P}^{-}_{[0,t]}}(\omega) \longrightarrow e_p, \text{ a.s.}$$

Gallavotti-Cohen Symmetry

Rate function of LDP for $e_p(t)(\omega)$:

$$I(z) = I(-z) - z.$$

$$\frac{\mathbf{P}\left(\frac{e_p(t)(\omega)}{t} = e_p\right)}{\mathbf{P}\left(\frac{e_p(t)(\omega)}{t} = -e_p\right)} \approx \exp(te_p).$$

Remark

Dissipative PDEs with Kicked Noise

Jaksic, Nersesyan, Pillet, Shirikyan (2015) *CPAM*
Jaksic, Nersesyan, Pillet, Shirikyan (2015) *CMP*

PROPOSITION (Chen & L. 14, Relation between real and complex Hermite polynomials)

$z = x + \mathrm{i}y$. Then

$$J_{m,l-m}(z) = \sum_{k=0}^{l} \mathrm{i}^{l-k} \sum_{r+s=k} \binom{m}{r}\binom{l-m}{s}(-1)^{l-m-s} H_k(x)H_{l-k}(y),$$

$$H_k(x)H_{l-k}(y) = \frac{\mathrm{i}^{l-k}}{2^l} \sum_{m=0}^{l} \sum_{r+s=m} \binom{k}{r}\binom{l-k}{s}(-1)^s J_{m,l-m}(z).$$

Thus, $\{J_{k,l}(z) : k+l=n\}$ and $\{H_k(x)H_l(y) : k+l=n\}$ generate the same linear subspace of $L^2_{\mathbb{C}}(\mathbb{C},\nu)$.

$$\overline{J_{m,n}(z,\rho)} = J_{n,m}(z,\rho).$$

$$E_\nu[J_{m,n}(z,\rho)^2] = E_\nu[J_{m,n}(z,\rho)\overline{J_{n,m}(z,\rho)}] = 0, \text{ if } m \neq n.$$

Proof

$$J_{m,l-m}(z) = \sum_{k=0}^{l} a_k H_k(x) H_{l-k}(y) + \mathrm{i} \sum_{k=0}^{l} b_k H_k(x) H_{l-k}(y)$$

$$\sum_{k=0}^{n} \bar{a}_k H_k(x) H_{n-k}(y) \stackrel{?}{=} H_n(x+y)$$

$$\sum_{k=0}^{n} \binom{n}{l} (\cos\theta)^k (\sin\theta)^{n-k} H_k(x) H_{n-k}(y) = H_n(x\cos\theta + y\sin\theta)$$

PROOF

$$J_{m,l-m}(z) = \sum_{k=0}^{l} \tilde{a}_k H_l(x\cos\theta_k + y\sin\theta_k) + \mathrm{i}\sum_{k=0}^{l} \tilde{b}_k H_l(x\cos\theta_k + y\sin\theta_k)$$

$$X, Y \text{ i.i.d.} \sim \mathcal{N}(0,1) \Rightarrow X\cos\theta + Y\sin\theta \sim \mathcal{N}(0,1)$$